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BOUNDARY BEHAVIOR OF $\bar\partial$ ON WEAKLY PSEUDO-CONVEX MANIFOLDS OF DIMENSION TWO

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1. Introduction

The problem of finding holomorphic functions in a domain which cannot be extended across the boundary, usually known as the Levi problem, seems to be intimately connected with various regularity properties of the operator $\bar\partial$. We will deal here with a complex manifold M with a smooth boundary, denoted by bM. Consider the following local version of the Levi problem: given $P \in bM$ find a holomorphic function in a neighborhood of P whose restriction to $\bar M$ vanishes exactly at P. A classical result states that whenever the Levi form is positive definite the problem has a solution, but if the Levi form has a negative eigenvalue or is identically zero in a neighborhood of P then the problem does not have a solution. This behaviour of the Levi form also controls the local (or more precisely the pseudo-local) regularity of the inhomogenous Cauchy-Riemann operator $\bar\partial$. It is natural to ask: what happens when the Levi form is positive semi-definite, vanishes at P, but not identically in a neighborhood. Here we establish some conditions for the solution of these problems.

We shall investigate the regularity properties of $\bar\partial$ by means of the $\bar\partial$-Neumann problem. On this occasion we do not wish to recall the history of this problem; we refer to [1] for a selfcontained treatment of the $\bar\partial$-Neumann problem as well as an historical discussion. However, since this paper is dedicated to Professor Spencer's sixtieth birthday, it is appropriate to point out that the $\bar\partial$-Neumann problem was first formulated by D. C. Spencer and that he pioneered several of its applications and generalizations to overdetermined systems. We shall impose some conditions on M and establish certain "subelliptic estimates" for the $\bar\partial$-Neumann problem. Among the consequences of such estimates are the following:

(i) Existence, regularity and pseudo-localness of a solution to the inhomogeneous Cauchy-Riemann equations. That is, whenever the above mentioned estimates hold, there is a unique solution of the equation $\bar\partial u = \alpha$ (where α is a $(0, 1)$-form satisfying the necessary compatibility condition), such that u is

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orthogonal to the holomorphic functions and u is smooth up to and including the boundary wherever α is smooth.

(ii) The operator $H: L_2(M) \rightarrow \mathscr{H}(M)$, which is orthogonal projection onto the space $\mathscr{H}(M)$ of square integrable holomorphic functions on M is pseudo-local. This, in particular, gives regularity properties at the boundary for the Bergman kernel function (see [7]).

(iii) Orthogonal decomposition and representation of cohomology classes by harmonic forms which are smooth on the closed manifold (i.e., including the boundary).

In the case of strongly pseudo-convex manifolds the appropriate subelliptic estimate and the above conclusions are established in [8]. For more general subelliptic estimates the above properties are proven in [11].

Here we are concerned with the problem of establishing subelliptic estimates where M is weakly (i.e., not strongly) pseudo-convex. First we wish to convince the reader that pseudo-convexity does not suffice to establish (i). We should distinguish between global and local regularity theorems; it is very likely that on any pseudo-convex M if α is in $C^\infty(\bar{M})$ then the solution in (i) is also in $C^\infty(\bar{M})$ (the writer has obtained results which point in this direction, and is currently working on the problem). However, the pseudo-local property, as in (i), is in general false. Suppose that in a neighborhood of the point $P \in bM$ (the boundary of M) the Levi form is identically zero. Then we can choose local holomorphic coordinates $z_1, \cdots, z_n$ with origin at P such that for some neighborhood U of P all points $Q \in U$, for which $z_n(Q) = 0$, lie in bM. Let $\alpha = \bar{\partial}(\rho/z_n) = \bar{\partial}\rho/z_n$, where $\rho \in C_0^\infty(U)$ and $\rho = 1$ in a neighborhood of P. Then (by a theorem in [4]) if the support of ρ is small enough there exists a function u on M such that $\bar{\partial}u = \alpha$. However, u cannot be smooth where α is, in particular, α is smooth outside the support of $\bar{\partial}\rho$ so that, if u is smooth there, then the holomorphic function $h = \rho/z_n - u$ has smooth boundary values when $\rho = 0$ and equals $1/z_n - u$ when $\rho = 1$. That this is impossible is easily established by the classical continuity method.

Thus we see that both the above version of the Levi problem and the problem posed by (i) do not have a solution if the Levi form is identically zero in a boundary neighborhood. In this paper we treat pseudo-convex manifolds of dimension 2; for these the Levi form is represented by a 1×1 matrix, i.e., a function. If $P \in bM$ we introduce the condition that P be of "type m" (m an integer, see § 2.3) which, roughly, tells us that the Levi form vanishes to order $m - 1$ in the holomorphic and anti-holomorphic directions tangent to bM. For example, consider $M \subset C^2$ given by: $|z_1|^{2p} + |z_2|^{2q} < 1$, let $P \in bM$ be defined by $z_1(P) = 0$ and $z_2(P) = 1$. Then P is a point of type $m = 2p - 1$ and our theorems apply in this case. However, the Levi problem is trivial for this example, the required function being $z_2 - 1$.

In § 7 we discuss the higher dimensional case as well as various problems which arise from our work.

2. The Levi invariants

Let M' be a complex manifold of complex dimension two, and M be a complex submanifold with C^∞ boundary bM, i.e., there exists a real valued C^∞ function r defined in a neighborhood of bM such that $dr \neq 0$ and $r(P) = 0$ if and only if $P \in bM$. We will choose r so that $r > 0$ outside of $\bar{M}$ and $r < 0$ in M. Let $P \in bM$ and let U be a coordinate neighborhood with holomorphic coordinates z_1 and z_2. A vector field L is said to be *holomorphic* if it can be written in the form

$$L = a^1\partial/\partial z_1 + a^2\partial/\partial z_2\ , \quad \text{where} \quad a^i \in C^\infty(U)\ . \tag{2.1}$$

A vector field L is called *tangential* if at each point of bM it is tangent to bM, i.e., if $L(r) = 0$ at $r = 0$. As usual we define, $\bar{L}$, the conjugate of L, by

$$\bar{L} = \bar{a}^1\partial/\partial\bar{z}_1 + \bar{a}^2\partial/\partial\bar{z}_2\ , \tag{2.2}$$

and if T_1 and T_2 are two vector fields we define the Lie bracket by $[T_1, T_2] = T_1T_2 - T_2T_1$. The Lie algebra generated by T_1 and T_2 over the C^∞ functions is the smallest module over the C^∞ functions closed under [,]; we denote it by $\mathscr{L}\{T_1, T_2\}$. $\mathscr{L}\{T_1, T_2\}$ is filtered, i.e.,

$$\mathscr{L}\{T_1, T_2\} = \bigcup_{k=0}^{\infty} \mathscr{L}_k\{T_1, T_2\}\ ,$$

where $\mathscr{L}_0\{T_1, T_2\}$ is the module spanned by T_1 and T_2, and $\mathscr{L}_{k+1}\{T_1, T_2\}$ is the module spanned by the elements of $\mathscr{L}_k\{T_1, T_2\}$ and the elements of the form $[A, T_i]$ with $A \in \mathscr{L}_k\{T_1, T_2\}$. We will set

$$\mathscr{L} = \mathscr{L}\{L, \bar{L}\}\ , \qquad \mathscr{L}_k = \mathscr{L}_k\{L, \bar{L}\}\ ,$$

where L is a holomorphic tangent vector at $P \in bM$ which is different from zero at P. Note that $\mathscr{L}$ and $\mathscr{L}_k$ evaluated at P do not depend on the choice of L.

2.3. Definition. $P \in bM$ is called of *finite type* if there exists $F \in \mathscr{L}$ such that $\langle(\partial r)_P, F_P\rangle \neq 0$. Here $\langle\ ,\ \rangle$ denotes the contraction between co-tangent vectors and tangent vectors, and the subscript P denotes the evaluation at P. We say P is of *type m* if P is of finite type and m is the least integer such that there is an element in $\mathscr{L}_m$ satisfying the above property.

Observe that if L is of type m then $\mathscr{L}_m$ contains all local vector fields tangent to bM. This follows from the fact that bM is 3-dimensional and that and $L, \bar{L}$ and F are independent, since $\langle\partial r, \bar{L}\rangle = 0$ and $\langle\partial r, L\rangle = \langle dr, L\rangle = 0$.

2.4. Proposition. *If L is a nonzero holomorphic tangential vector field in a neighborhood of $P \in bM$, and $(i_0i_1\cdots i_m)$ is an $(m+1)$-tuple of zeros and ones, we define the vector fields $L^{(i_0\cdots i_m)}$ inductively by $L^{(0)} = L, L^{(1)} = \bar{L}$ and $L^{(i_0\cdots i_m)} = [L^{(i_m)}, L^{(i_0\cdots i_{m-1})}]$. Then P is of type m if and only if for some $(i_0\cdots i_m)$ we have $\langle(\partial r)_P, L_P^{(i_0\cdots i_m)}\rangle \neq 0$.*

Proof. It will suffice to show that the $\{L^{(i_0\cdots i_m)}\}$ form a basis of the Lie algebra generated by L and $\bar{L}$. For then any element F can be written as a finite sum of the form

$$F = \sum c_{i_0\cdots i_m} L^{(i_0\cdots i_m)} .$$

Thus

$$\langle(\partial r)_P, F_P\rangle = \sum c_{i_0\cdots i_m}(P)\langle(\partial r)_P, L_P^{(i_0\cdots i_m)}\rangle ,$$

and that this is different from zero implies some term in the above sum is different from zero.

To show that $\{L^{(i_0\cdots i_m)}\}$ is a basis it suffices to show that an element of the form $[L^{i_0\cdots i_r}, L^{j_0\cdots j_s}]$ is a linear combination of the $\{L^{(i_0\cdots i_m)}\}$. To prove this we proceed by induction on r. By the Jacobi identity we have

$$\begin{aligned}[L^{(i_0\cdots i_r)}, L^{(j_0\cdots j_s)}] &= [[L^{(i_r)}, L^{(i_0\cdots i_{r-1})}], L^{(j_0\cdots j_s)}] \\ &= [L^{(i_r)}, [L^{(i_0\cdots i_{r-1})}, L^{(j_0\cdots j_s)}]] - [L^{(i_0\cdots i_{r-1})}, L^{(j_0\cdots j_s i_r)}] .\end{aligned}$$

The induction hypothesis implies that the last term can be written as asserted and that

$$\begin{aligned}[L^{(i_r)}, L[^{(i_0\cdots i_{r-1})}, L^{(j_0\cdots j_s)}]] &= [L^{(i_r)}, \sum a_{k_0\cdots k_m} L^{(k_0\cdots k_m)}] \\ &= \sum \{L^{(i_r)}(a_{k_0\cdots k_m})L^{(k_0\cdots k_m)} + a_{k_0\cdots k_m} L^{(k_0\cdots k_m i_r)}\} ,\end{aligned}$$

which completes the proof.

2.5. Proposition. *If $P \in bM$ is of type m, and g is a differentiable function defined in a neighborhood of P, then for $s \leq m$,*

$$\langle(\partial r)_P, (gL)_P^{(i_0\cdots i_s)}\rangle = (g(P))^{m-k_s+1}(\overline{g(P)})^{k_s}\langle(\partial r)_P, L_P^{(i_0\cdots i_s)}\rangle , \tag{2.6}$$

where $k_s = \sum_{v=0}^{s} i_v$.

Proof. The following formula is easily established by induction on s:

$$(gL)^{(i_0\cdots i_s)} = (g)^{s-k_s+1}(\bar{g})^{k_s} L^{(i_0\cdots i_s)} + \sum_{p<s} c_{j_0\cdots j_p} L^{(j_0\cdots j_p)}) , \tag{2.7}$$

from which (2.6) follows when $s \leq m$.

2.8. Proposition. *If $P \in bM$ is of type m, and L is a holomorphic tangential vector field in a neighborhood of P, then by setting*

$$\lambda^{i_0\cdots i_p} = \langle\partial r, L^{(i_0\cdots i_p)}\rangle , \qquad p \geq 1 , \tag{2.9}$$

we have $\lambda^{00} = \lambda^{11} = 0$ and

$$\lambda^{10} = -\lambda^{01} = \langle\partial\bar{\partial} r, L \wedge \bar{L}\rangle . \tag{2.10}$$

Furthermore, if $2 \leq p \leq m$, *then*

$$\lambda^{10\, i_2 \cdots i_p}(p) = [(L)^{p-k-1}(\bar{L})^k \lambda^{10}]_P \,, \tag{2.11}$$

where $k = \sum_{v=2}^{P} i_v$.

Proof. Since $\partial\bar\partial r = -d\partial r$, we have by a classical expression for the exterior derivative:

$$\begin{aligned}\langle \partial\bar\partial r, L \wedge \bar{L}\rangle &= -\langle d\partial r, L \wedge \bar{L}\rangle \\ &= -L(\langle \partial r, \bar{L}\rangle) + \bar{L}(\langle \partial r, L\rangle) + \langle \partial r, [L, \bar{L}]\rangle \,.\end{aligned}$$

Then (2.10) follows since $\langle \partial r, \bar{L}\rangle = 0$ and $\langle \partial r, L\rangle = \langle dr, L\rangle = L(r) = 0$.

If $L_P = 0$ then (2.11) is trivial. So suppose that $L_P \neq 0$, set $L_1 = L$ and let L_2 be a holomorphic vector field such that $L_2(r) = 1$. Then $\langle \partial r, L_2\rangle = 1$. Note that any tangential vector field can be written as a combination of L_1, $\bar{L}_1$ and $L_2 - \bar{L}_2$. Thus we have

$$L_1^{(10\, i_2 \cdots i_p)} = \lambda^{10\, i_2 \cdots i_p}(L_2 - \bar{L}_2) + \sigma^{10\, i_2 \cdots i_p} L_1 + \mu^{10\, i_2 \cdots i_p} \bar{L}_1 \,. \tag{2.12}$$

Let $\mathscr{I}_1$ be the ideal generated by λ^{10}, and $\mathscr{I}_p$ for $p > 1$ be the ideal generated by $\mathscr{I}_{p-1}$ and the $\lambda^{10\, i_2 \cdots i_p}$. Then

$$\lambda^{10\, i_2 \cdots i_p} = L^{(i_p)} L^{(i_{p-1})} \cdots L^{(i_2)}(\lambda^{10}) \qquad (\text{mod } \mathscr{I}_{p-1}) \tag{2.13}$$

follows by induction on p, since

$$\begin{aligned}L_1^{(10 \cdots i_{p+1})} = \{&L_1^{(i_{p+1})}(\lambda^{10\, i_2 \cdots i_p}) + \lambda^{10\, i_2 \cdots i_p}\theta^{i_{p+1}} + \lambda^{0\, i_{p+1}}\sigma^{10\, i_2 \cdots i_p} \\ &+ \lambda^{1\, i_{p+1}}\mu^{10\, i_2 \cdots i_p}\}(L_2 - \bar{L}_2) \qquad (\text{mod } \mathscr{L}_0)\end{aligned}$$

where $\theta^{i_{p+1}}$ is defined by

$$[L_1^{i_{p+1}}, L_2 - \bar{L}_2] = \theta^{i_{p+1}}(L_2 - \bar{L}_2) \qquad (\text{mod } \mathscr{L}_0) \,.$$

Since, whenever $p \leq m$, the elements of $\mathscr{I}_{p-1}$ vanish at P, we have proved that

$$\lambda^{10\, i_2 \cdots i_p}(P) = [L^{(i_p)} \cdots L^{(i_2)} \lambda^{10}]_P \,.$$

To conclude the proof of (2.11) it suffices to show that

$$L_1^{(i_p)} \cdots L_1^{(i_{v+1})}[L_1^{(i_v)}, L_1^{(i_{v-1})}] L_1^{(i_{v-2})} \cdots L_1^{(i_2)} \lambda^{10} = 0 \qquad (\text{mod } \mathscr{I}_{p-1}) \,.$$

Since

$$[L_1^{(i_v)}, L_1^{(i_{v-1})}] = \lambda^{i_{v-1} i_v}(L_2 - \bar{L}_2) + \sigma^{i_{v-1} i_v} L_1 + \mu^{i_{v-1} i_v} \bar{L}_1 \,,$$

the desired result is easily obtained by induction.

2.14. Definition. M is *pseudo-convex* if on bM we have $\langle \partial\bar{\partial} r, L \wedge \bar{L}\rangle \geq 0$ where L is a nonzero tangential holomorphic vector field.

2.15. Definition. If M is pseudo-convex, and $P \in bM$ is of type m, then we say that bM is *pseudo-convex of order m at P*.

2.16. Definition. If P is of type m, and

$$\sum_{s+t=m-1} \frac{1}{(s+1)!\,(t+1)!}[(L)^s(\bar{L})^t \lambda^{10}]_P > 0 \tag{2.17}$$

whenever $L \neq 0$ is a holomorphic tangential vector field in a neighborhood of P (and when λ^{10} is given by (2.10)), then we say that P is *of strict type m*.

If M is pseudo-convex of order 1 at P then M is strongly pseudo-convex at P in the classical sense.

3. The local Levi problem

If $P \in bM$, the problem of finding a holomorphic function in M which cannot be continued past P is called the Levi problem. In this section we prove the following result which yields a local solution to the Levi problem (by taking the reciprocal of the function h).

3.1. Theorem. *If M is pseudo-convex, and $P \in bM$ is pseudo-convex of type m, then m is odd.*[1] *If P is of strict type m, then there exist a neighborhood U of P and a holomorphic function h such that*

$$\{Q \in U \cap \bar{M} \mid h(Q) = 0\} = \{P\}\ . \tag{3.2}$$

In fact, in terms of local holomorphic coordinates the function h defined by

$$h(z_1, z_2) = \sum_{s+t\leq m+1} \frac{1}{s!\,t!}[(\partial/\partial z_1)^s(\partial/\partial z_2)^t r]_P (z_1 - z_1(P))^s (z_2 - z_2(P))^t \tag{3.3}$$

satisfies (3.2).

Proof. By an affine change of coordinates we can construct coordinates z_1' and z_2' such that

$$z_1'(P) = z_2'(P) = (\partial r/\partial z_1')_P = (\partial r/\partial \bar{z}_1')_P = (\partial r/\partial y_2')_P = 0\ , \tag{3.4}$$

and $(\partial r/\partial x_2')_P = 1$, where $z_1' = x_1' + iy_1'$ and $z_2' = x_2' + iy_2'$.

Expanding r in a Taylor series we have

$$r(z') = \operatorname{Re} h'(z') + \psi(z') + 0(|z'|^{m+2})\ , \tag{3.5}$$

where $h'(z')$ is the function defined by (3.3) in terms of the coordinates z', and

[1] **Added in proof.** This part of the theorem follows immediately from Propositiion 2.4 in I of [13].

$\psi(z')$ is a polynomial in $z_1', z_2', \bar z_1', \bar z_2'$ of degree $m+1$ which does not contain any "pure" terms (i.e., each term contains $z_i'\bar z_j'$ as a factor). Observe that the functions z_1' and h' are independent in a neighborhood of 0 since by (3.4) and (3.5)

$$(3.6)\qquad (\partial h'/\partial z_1')_0 = 0\ , \qquad (\partial h'/\partial z_2')_0 = (\partial r/\partial x_2')_0 = 1\ .$$

Thus we can introduce holomorphic coordinates z_1, z_2 defined by

$$(3.7)\qquad z_1 = z_1'\ , \qquad z_2 = h'\ ,$$

and as usual we set $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. In terms of these coordinates the expansion (3.5) becomes

$$(3.8)\qquad r(z_1, z_2) = x_2 + \theta(z_1, z_2) + 0(|z_1|^{m+2} + |z_2|^{m+2})\ ,$$

and thus the function h defined by (3.3) is z_2, and θ is given by

$$(3.9)\qquad \theta(z_1, z_2) = \sum_{2\le s_1+s_2+t_1+t_2\le m+1} a_{s_1s_2t_1t_2} z_1^{s_1} z_2^{s_2} \bar z_1^{t_1} \bar z_2^{t_2}\ ,$$

where

$$a_{s_1s_2t_1t_2} = \begin{cases} 0\ , & \text{if } s_1 + s_2 = 0\ ,\\ 0\ , & \text{if } t_1 + t_2 = 0\ ,\\ \dfrac{1}{s_1!\, s_2!\, t_1!\, t_2!}\left[\left(\dfrac{\partial}{\partial z_1}\right)^{s_1}\left(\dfrac{\partial}{\partial z_2}\right)^{s_2}\left(\dfrac{\partial}{\partial \bar z_1}\right)^{t_1}\left(\dfrac{\partial}{\partial \bar z_2}\right)^{t_2} r\right]\ , & \text{otherwise.}\end{cases}$$

We will prove the theorem by showing that $\theta(z_1, 0)$ is a homogeneous polynomial of degree $m+1$ and that

$$(3.10)\qquad [\partial^2\theta/\partial z_1\partial \bar z_1]_{z_2=0} \ge 0\ .$$

We will do this by showing that for a suitable holomorphic tangential vector field L we have

$$(3.11)\qquad [(\partial/\partial z_1)^{s+1}(\partial/\partial \bar z_1)^{t+1} r] = [L^s \bar L^t \lambda^{10}]_0$$

when $s + t \le m - 1$, where λ^{10} is defined by (2.10). Then

$$(3.12)\qquad \theta(z_1, 0) = \sum_{s+t=m-1} \frac{1}{(s+1)!\,(t+1)!}[L^s \bar L^t \lambda^{10}]_0\, z_1^{s+1}\bar z_1^{t+1}\ .$$

From this, (3.10) will be deduced after showing that the left hand side of (3.10) can be identified with the restriction of λ^{10} to a real 2-dimensional surface in bM.

It follows from (3.11) that m is odd, and if P is of strict type m then (3.12) implies that $\theta(z_1, 0) \geq \text{const.}\, |z_1|^{m+1}$, so that when $z_2 = 0$ we have $r(z_1, 0) \geq \text{const.}\, |z_1|^{m+1}$ if $|z_1|$ is small enough, which proves (3.2) since $h = z_2$.

These facts are proven in the lemmas given below.

3.13. Lemma. *In terms of the coordinates constructed above we have*

$$[\partial r/\partial z_2]_0 = 1 \quad \text{and} \quad [(\partial/\partial z_1)^s(\partial/\partial z_2)^t r]_0 = 0 \tag{3.14}$$

for $s + t \leq m + 1$ and $s > 0$ when $t = 1$.

Proof. This follows immediately from (3.8).

3.15. Definition. Let $\mathcal{J}_k$ be the ideal of germs of C^∞ functions at the origin generated by $\{(\partial/\partial z_1)^s(\partial/\partial \bar{z}_1)^t r\}$ with $s + t \leq k$.

3.16. Lemma. *If $P \in bM$ is of type m, then for $k \leq m$ the elements of $\mathcal{J}_k$ evaluated at P vanish. Furthermore, if in terms of the coordinate system introduced above we define the holomorphic vector field L, which is tangential to bM, by*

$$L = (\partial r/\partial z_2)\partial/\partial z_1 - (\partial r/\partial z_1)\partial/\partial z_2\ , \tag{3.17}$$

and if λ^{10} is given by (2.10), *then whenever $s + t \leq m - 1$ we have*

$$[L^s \bar{L}^t \lambda^{10}] = [(\partial/\partial z_1)^{s+1}(\partial/\partial \bar{z}_1)^{t+1} r]_0\ . \tag{3.18}$$

Proof. First observe that $L(\mathcal{J}_k) \subset \mathcal{J}_{k+1}$ and $\bar{L}(\mathcal{J}_k) \subset \mathcal{J}_{k+1}$. Next, we have

$$\begin{aligned}\lambda^{10} &= \langle \partial\bar{\partial} r, L \wedge \bar{L}\rangle \\ &= \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1}\left|\frac{\partial r}{\partial z_2}\right|^2 + \frac{\partial^2 r}{\partial z_2 \partial \bar{z}_2}\left|\frac{\partial r}{\partial z_1}\right|^2 - \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_2}\frac{\partial r}{\partial z_2}\frac{\partial r}{\partial \bar{z}_1} \\ &\quad - \frac{\partial^2 r}{\partial z_2 \partial \bar{z}_1}\frac{\partial r}{\partial z_1}\frac{\partial r}{\partial \bar{z}_2}\ ,\end{aligned} \tag{3.19}$$

so that

$$\lambda^{10} = \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1}\left|\frac{\partial r}{\partial z_2}\right|^2 \qquad (\text{mod } \mathcal{J}_1)\ . \tag{3.20}$$

By induction we see that

$$L^s \bar{L}^t \lambda^{10} = \left\{\left(\frac{\partial}{\partial z_1}\right)^{s+t}\left(\frac{\partial}{\partial \bar{z}_1}\right)^{t+1} r\right\}\left(\frac{\partial r}{\partial z_2}\right)^{s+1}\left(\frac{\partial r}{\partial \bar{z}_2}\right)^{t+1} \qquad (\text{mod } \mathcal{J}_{s+t+1})\ . \tag{3.21}$$

Since 0 is a point of type m, by (2.11) we have

$$[L^s \bar{L}^t \lambda^{10}]_0 = 0 \qquad \text{if } s + t < m - 1\ . \tag{3.22}$$

Since the elements of $\mathcal{J}_1$ evaluated at 0 vanish, from (3.20) we obtain (3.18) for $s = t = 0$. It is clear from (3.21) that if the elements of $\mathscr{I}_k$ vanish at 0 then

$$[(\partial/\partial z_1)^{s+1}(\partial/\partial \bar z_1)^{t+1} r]_0 = 0 \qquad \text{for } s + t \le k - 1 \le m - 1 \ .$$

On the other hand this, Lemma 3.13 and (3.22) imply that the elements of $\mathscr{I}_k$ vanish at 0 for $k \le m$, which concludes the proof of the lemma.

3.23. Lemma. *Under the same assumptions as above if f is a function defined in a neighborhood of* 0, *L is given by* (3.17), *and*

$$[L^s \bar L^t f]_0 = 0 \qquad \textit{for } s + t \le p \le m - 1 \ , \tag{3.24}$$

then

$$[L^s \bar L^t f]_0 = [(\partial/\partial z_1)^s(\partial/\partial \bar z_1)^t f]_0 \qquad \textit{for } s + t \le p + 1 \ . \tag{3.25}$$

Proof. By definition we have

$$Lf = (\partial r/\partial z_2)\partial f/\partial z_1 \pmod{\mathscr{I}_1} \ , \qquad \bar L f = (\partial r/\partial \bar z_2)\partial f/\partial \bar z_1 \pmod{\mathscr{I}_1} \ .$$

Let $\mathscr{F}_1 = \mathscr{I}_1$, and let $\mathscr{F}_k$ be the ideal generated by $\mathscr{I}_k$ and the elements $\{(\partial/\partial z_1)^s(\partial/\partial \bar z_1)^t f\}$ with $s + t \le k - 1$. Then $L(\mathscr{F}_k) \subset \mathscr{F}_{k+1}$ and $\bar L(\mathscr{F}_k) \subset \mathscr{F}_{k+1}$. Thus by induction we obtain

$$L^s \bar L^t f = (\partial r/\partial z_2)^s(\partial r/\partial \bar z_2)^t\{(\partial/\partial z_1)^s(\partial/\partial \bar z_1)^t f\} \pmod{\mathscr{F}_{s+t}} \ .$$

It then follows by induction on k (for $k \le p + 1$) that $\mathscr{F}_k$ vanishes at 0 and that (3.25) holds for $k = s + t \le p + 1$.

The proof of Theorem 3.1 is concluded with the following lemma.

3.26. Lemma. *If M is pseudo-convex, and $P \in bM$ is of type m, then* (3.12) *and the inequality* (3.10) *hold.*

Proof. In a neighborhood of 0 we define the surface S by the equations $r = 0$ and $y_2 = 0$. Let w be the restriction of z_1 to S. Then w and $\bar w$ are local coordinates in a neighborhood of S. In a neighborhood of 0 in M' we define the vector field T by:

$$T = \partial/\partial z_1 - (\partial r/\partial z_1)(\partial r/\partial x_2)^{-1}\partial/\partial x_2 \ . \tag{3.27}$$

Then the restriction of T to S is $\partial/\partial w$, and the restriction of $\bar T$ to S is $\partial/\partial \bar w$. Further we have for any function f

$$(T)^s(\bar T)^t f = (\partial/\partial z_1)^s(\partial/\partial \bar z_1)^t f \pmod{\mathscr{I}_{s+t}} \ . \tag{3.28}$$

Since the elements of $\mathscr{I}_k$ vanish at 0 when $k \le m$, and since $0 \in S$ we conclude

$$[(T)^s(\bar T)^t f]_0 = [(\partial/\partial w)^s(\partial/\partial \bar w)^t f]_0 = [(\partial/\partial z_1)^s(\partial/\partial \bar z_1)^t f]_0 \tag{3.29}$$

for $s + t \le m$.

Setting $f = \lambda^{10}$, then Proposition 2.8 shows that the hypotheses of Lemma 2.23 are fulfilled with $p = m - 2$, and hence we have

$$(3.30)\qquad \begin{aligned}[(L)^s(\bar{L})^t\lambda^{10}]_0 &= [(\partial/\partial w)^s(\partial/\partial \bar{w})^t\lambda^{10}]_0 \\ &= [(\partial/\partial z_1)^s(\partial/\partial \bar{z}_1)^t\lambda^{10}]_0 \qquad \text{for } s + t < m .\end{aligned}$$

Combining this with (3.18) we obtain (3.12).

Expanding the restriction of λ^{10} to S in a Taylor series we obtain

$$(3.31)\qquad \lambda^{10} = \sum_{s+t=m-1} \frac{1}{s!\,t!}[(L)^s(\bar{L})^t\lambda^{10}]_0 w^s\bar{w}^t + 0(|w|^m) .$$

Since $\lambda^{10} \geq 0$ on S, the above sum must be nonnegative and m must be odd. Hence (3.10) follows immediately, and this concludes the proof of the lemma and also of Theorem 3.1.

4. The basic estimate

Suppose M' has a hermitian metric. We will denote by $\langle\ ,\ \rangle$ the inner product under this metric in the tangent space as well as the inner product induced in the space of forms, and the corresponding norms will be denoted by $|\ \ |$. Define the inner product on forms by

$$(\varphi, \psi) = \int_M \langle\varphi, \psi\rangle dV \quad \text{and} \quad \|\varphi\|^2 = (\varphi, \varphi) ,$$

where dV, the volume element, is the unique real $(2, 2)$-form of length one which agrees with the natural orientation of M'.

Choose r so that $|\partial r| = 1$ on bM. If $P \in bM$, then in a neighborhood U of P we choose ω^1 and ω^2 to be an orthonormal basis for the $(1, 0)$-forms at each point of U and such that $\omega^2 = f\partial r$ (with $f = 1$ on bM). Let L_1, L_2 be the corresponding dual basis. Then

$$0 = \langle\omega^2, L_1\rangle = f\langle\partial r, L_1\rangle = f\langle dr, L_1\rangle = fL_1(r) ,$$

so that L_1 is a tangential holomorphic vector field. Now if φ is a $(0, 1)$-form, then in U we have

$$(4.1)\qquad \varphi = \varphi_1\bar{\omega}^1 + \varphi_2\bar{\omega}^2 ,$$

$$(4.2)\qquad \bar{\partial}\varphi = (\bar{L}_1\varphi_2 - \bar{L}_2\varphi_1)\bar{\omega}^1 \wedge \bar{\omega}^2 + \cdots ,$$

where the dots stand for terms containing undifferentiated combinations of the φ_1 and φ_2. The formal adjoint of $\bar{\partial}$ is given in U by:

$$(4.3)\qquad \vartheta\varphi = -L_1\varphi_1 - L_2\varphi_2 + \cdots ,$$

where again the dots stand for a linear combination of φ_1 and φ_2.

Let $\mathscr{B}$ denote the space of (0, 1)-forms with compact support in U such that $\langle\varphi, \bar\partial r\rangle = 0$ on bM, i.e., in terms of the representation (4.1) we have $\varphi_2 = 0$ on bM.

4.4. Proposition. *If M is pseudo-convex, then there exists $C > 0$ such that*

$$\int_{bM} \lambda^{10}|\varphi_1|^2 dS + \|L_1\varphi_1\|^2 \sum_{i=1}^{2} \|L_i\varphi_2\|^2 + \sum_{i,j=1} \|\bar L_i\varphi_j\|^2 \tag{4.5}$$
$$\leq C(\|\bar\partial\varphi\|^2 + \|\vartheta\varphi\|^2 + \|\varphi\|^2) , \qquad \text{for all } \varphi \in \mathscr{B} ,$$

where λ^{10} is the Levi form on the boundary, i.e., by (2.9),

$$[L_1, \bar L_1] = \lambda^{10}(L_2 - \bar L_2) + gL_1 - \bar g\bar L_1 .$$

Proof. From (4.2) we obtain

$$\|\bar\partial\varphi\|^2 = \|\varphi\|_{\bar z}^2 - \sum (\bar L_i\varphi_j, \bar L_j\varphi_i) + 0(\|\varphi\|_{\bar z}\|\varphi\| + \|\varphi\|^2) , \tag{4.6}$$

where $\|\varphi\|_{\bar z}^2 = \sum \|\bar L_i\varphi_j\|^2$. From (4.3) we have

$$\|\vartheta\varphi\|^2 = \sum (L_i\varphi_i, L_j\varphi_j) + 0(\|\varphi\|_{\bar z}\|\varphi\| + \|\varphi\|^2) , \tag{4.7}$$

since the error term on the right estimates terms of the form $(L_i\varphi_i, \cdots)$; by integration by parts there are no boundary terms since L_1 is tangential and $\varphi_2 = 0$ on bM. Now setting

$$[L_i, \bar L_j] = \sum c_{ij}^k L_k + \sum d_{ij}^k \bar L_k ,$$

where $c_{11}^2 = \lambda^{10}$, and by integration by parts we obtain

$$\begin{aligned} -\sum (\bar L_i\varphi_j, \bar L_j\varphi_i) &= \sum (L_j\bar L_i\varphi_j, \varphi_i) + 0(\|\varphi\|_{\bar z}\|\varphi\| + \|\varphi\|^2) \\ &= -\sum (L_j\varphi_j, L_i\varphi_i) + \sum (c_{ji}^2 L_2\varphi_j, \varphi_i) + 0(\|\varphi\|_{\bar z}\|\varphi\| + \|\varphi\|^2) \\ &= -\|\vartheta\varphi\|^2 + \sum \int_{bM} c_{ji}^2\varphi_j\bar\varphi_i dS + 0(\|\varphi\|_{\bar z}\|\varphi\| + \|\varphi\|^2) . \end{aligned}$$

The second term on the right equals $\int_{bM} \lambda^{10}|\varphi_1|^2 dS$ since $\varphi_2 = 0$ on bM. Substituting this in (4.6) and using the inequality

$$|ab| \leq \text{small const. } |a|^2 + \text{large const. } |b|^2$$

we obtain

$$\|\varphi\|_{\bar z}^2 + \int_{bM} \lambda^{10}|\varphi_1|^2 dS \leq \text{const. } (\|\bar\partial\varphi\|^2 + \|\vartheta\varphi\|^2 + \|\varphi\|^2) . \tag{4.8}$$

Furthermore, since $\varphi_2 = 0$ on bM we have

$$\|L_i\varphi_2\|^2 = -(\bar{L}_i L_i \varphi_2, \varphi_2) + 0(\|\varphi\|_{\bar z}\|\varphi\|) \tag{4.9}$$
$$= \|\bar{L}_i\varphi_2\|^2 + 0(\|\varphi\|_{\bar z}\|\varphi\| + \|\varphi\|^2) \le \text{const.}\ (\|\varphi\|_{\bar z}^2 + \|\varphi\|^2)\ .$$

Finally,

$$\begin{aligned}\|L_1\varphi_1\|^2 &= -(\bar{L}_1 L_1\varphi_1, \varphi_1) + 0(\|\varphi\|_{\bar z}\|\varphi\|) \\ &= \|\bar{L}_1\varphi_1\|^2 + (\lambda^{10} L_2\varphi_1, \varphi_1) + 0(\|\varphi\|_{\bar z}\|\varphi\|) \\ &\le \int_{bM} \lambda^{10}|\varphi_1|^2 dS + \text{const.}\ (\|\varphi\|_{\bar z}^2 + \|\varphi\|^2)\ .\end{aligned} \tag{4.10}$$

The desired inequality (4.5) is then obtained by combining (4.8), (4.9) and (4.10).

5. The tangential Sobolev norms

In a neighborhood U of $P \in bM$ we introduce boundary coordinates (x', x_4) where $x' = (x_1, x_2, x_3)$ are coordinates in $(bM) \cap U$ and $x_4 = r$. We call x' the *tangential coordinates* and x_4 the normal coordinate. For $u \in C_0^\infty(U)$ we define the tangential Fourier transform by

$$\tilde{u}(\xi', x_4) = \int e^{-ix'\cdot\xi'} u(x', x_4) dx'\ , \tag{5.1}$$

where $\xi' = (\xi_1, \xi_2, \xi_3)$, $x'\cdot\xi' = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$ and $dx' = dx_1 dx_2 dx_3$ For each $s \in \boldsymbol{R}$ we define the partial Sobolev s-norm by

$$|||u|||_s^2 = \int_{\boldsymbol{R}^3}\int_{-\infty}^0 (1 + |\xi'|^2)^s |\tilde{u}(\xi', x_4)|^2 d\xi' dx_4\ . \tag{5.2}$$

For each s we define the operator Λ^s ("the $s/2$ power of the tangential laplacian") by

$$\widetilde{\Lambda^s u}(\xi', x_4) = (1 + |\xi'|^2)^{s/2}\tilde{u}(\xi', x_4)\ . \tag{5.3}$$

Then we have, by the Plancherel theorem,

$$|||u|||_s = \|\Lambda^s u\|\ . \tag{5.4}$$

5.5. Definition. A *tangential differential operator* T of order m is a differential operator which can be expressed in the form

$$Tu = \sum_{\alpha_1+\alpha_2+\alpha_3\le m} a_{\alpha_1\alpha_2\alpha_3}\frac{\partial^{\alpha_1+\alpha_2+\alpha_3}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\ . \tag{5.6}$$

We define an algebra $\mathscr{T}$ of "*tangential pseudo-differential operators*" to be the algebra generated by the tangential differential operators and the Λ^s under the operations of composition, addition and L_2-adjoints.

5.7. Definition. $T \in \mathscr{T}$ is of order r if for each s there exists a constant C_s such that

$$|||Tu|||_s \leq C_s|||u|||_{s+r}$$

for all $u \in C_0^\infty(U \cap \bar{M})$.

The following proposition lists the properties of the algebra $\mathscr{T}$ which we shall need. The proof is exactly the same as the proof of the corresponding properties of ordinary pseudo-differential operators; see for example [12].

5.8. Proposition. *Every $T \in \mathscr{T}$ has finite order, Λ^s has order s, and P given by* (5.6) *has order m. If T_1 and $T_2 \in \mathscr{T}$ are of order m_1 and m_2 respectively, then T_1T_2 is of order $m_1 + m_2$, $[T_1, T_2] = T_1T_2 - T_2T_1$ is of order $m_1 + m_2 - 1$, and T_1^* (the L_2-adjoint of T_1) is of order m_1.*

The type of argument used in the following lemma was discovered independently by Radkevitch [13] and the author [7]. Sharper results can be obtained from a theorem of Hörmander which are discussed in § 7.

5.9. Lemma. *If $P \in bM$ is of type m, then there exist a neighborhood U of P and a constant $C > 0$ such that*

$$\sum_{j=1}^{4} \left|\left|\left| \frac{\partial u}{\partial x_j} \right|\right|\right|_{\varepsilon-1} \leq C(\|L_1u\| + \|\bar{L}_1u\| + \|\bar{L}_2u\| + \|u\|) \tag{5.10}$$

for $0 < \varepsilon \leq 2^{-m}$ and all $u \in C_0^\infty(U \cap \bar{M})$.

Proof. Since $\partial/\partial x_j$ is a linear combination of $L_1, \bar{L}_1, L_2$ and $\bar{L}_2$, it suffices to show that $|||L_2u|||_{\varepsilon-1}$ is bounded by the right hand side of (5.10). Now if we choose U small enough, the assumption that P is of type m implies that L_2 is a linear combination of $L_1^{(i_0\cdots i_m)}, L_1, \bar{L}_1$ and $\bar{L}_2$, so it suffices to show that $|||L_1^{(i_0\cdots i_m)}u|||_{\varepsilon-1}$ is bounded by the right hand side of (5.10). Denoting by T^s an element of $\mathscr{T}$ of order s we have

$$\begin{aligned} |||L_1^{(i_0\cdots i_m)}u|||_{\varepsilon-1}^2 &= (L_1^{(i_0\cdots i_m)}u, T^{2\varepsilon-1}u) \\ &= ([L_1^{(i_m)}, L^{(i_0\cdots i_{m-1})}]u, T^{2\varepsilon-1}u) \\ &= (L_1^{(i_m)}L_1^{(i_0\cdots i_{m-1})}u, T^{2\varepsilon-1}u) - (L_1^{(i_0\cdots i_{m-1})}L_1^{(i_m)}u, T^{2\varepsilon-1}u) \ . \end{aligned} \tag{5.11}$$

Since $(L_1^{(i_m)})^* = -\bar{L}_1^{(i_m)} + T^0$, and $[\bar{L}^{(i_m)}, T^{2\varepsilon-1}]$ is of order $2\varepsilon - 1$, we have

$$\begin{aligned} &(L_1^{(i_m)}L_1^{(i_0\cdots i_{m-1})}u, T^{2\varepsilon-1}u) \\ &\quad = -((T^{2\varepsilon-1})^*L_1^{(i_0\cdots i_{m-1})}u, \bar{L}_1^{(i_m)}u) + (L_1^{(i_0\cdots i_{m-1})}u, T^{2\varepsilon-1}u) \ , \end{aligned}$$

and since $(T^{2\varepsilon-1})^*$ is of order $2\varepsilon - 1$ we see that the first term on the right of (5.11) is bounded by

$$\text{(5.12)} \qquad \text{const.}\ \{|||L_1^{(i_0\cdots i_{m-1})}u|||^2_{2\varepsilon-1} + \cdots\}\,,$$

where the dots stand for the right side of (5.10). Similarly, since $(L_1^{(i_0\cdots i_{m-1})})^* = -\bar{L}_1^{(i_0\cdots i_{m-1})} + T^0$, we can bound the second term on the right of (5.11) by (5.12), and hence we obtain

$$\text{(5.13)} \qquad |||L_1^{(i_0\cdots i_m)}u|||_{\varepsilon-1} \leq \text{const.}\ \{|||L_1^{(i_0\cdots i_{m-1})}u|||_{2\varepsilon-1} + \cdots\}\,.$$

Applying the same argument to the first term on the right of (5.13) m times we obtain the desired estimate provided $\varepsilon \leq 2^{-m}$.

Combining the above lemma with Proposition 4.4 we obtain the following result.

5.14. Theorem. *If M is pseudo-convex, and $P \in bM$ is of type m, then there exist a neighborhood U of P and a constant $C > 0$ such that*

$$\text{(5.15)} \qquad |||D\varphi|||^2_{2^{-m}-1} \leq C(\|\bar{\partial}\varphi\|^2 + \|\vartheta\varphi\|^2 + \|\varphi\|^2)$$

for all $\varphi \in C_0^\infty(U \cap \bar{M}) \cap \mathscr{D}$, where we define

$$\text{(5.16)} \qquad |||D\varphi|||^2_s = \sum_{j=1}^{4} |||\partial\varphi_i/\partial x_j|||^2_s\,.$$

6. The $\bar{\partial}$-Neumann problem, boundary regularity, and the global Levi problem

In this section we summarize some of the consequences of estimates of the type:

$$\text{(6.1)} \qquad |||D\varphi|||^2_{s-1} \leq C(\|\bar{\partial}\varphi\|^2 + \|\vartheta\varphi\|^2 + \|\varphi\|^2)\,,$$

with $0 < s < 1$ for all $\varphi \in C_0^\infty(U \cap \bar{M}) \cap \mathscr{D}$, as in Theorem 5.4. The case $s = 1/2$ was treated in [8], and the general case in [11].

6.2. Definition. A manifold M with smooth boundary and compact $\bar{M}$ is called of *type m*, if every $P \in bM$ is of type m_P, and $m = \max_{P \in bM} m_P$. Note that the maximum in the above definition exists since if $P \in bM$ is of type m_P then there is a neighborhood of U of P such that $m_Q \leq m_P$ whenever $Q \in U \cap bM$, and since bM is compact we can take the maximum over only a finite number of such neighborhoods. Also note that from Theorem 5.14 it follows that if M is pseudo-convex and of type m then (6.1) holds with $s = 2^{-m}$. In § 8 we show that (6.1) holds with $s < 1/(m + 1)$.

6.3. Theorem. *If M is a manifold with smooth boundary and compact $\bar{M}$, and if for every $P \in bM$ there exists a neighborhood U such that* (6.1) *holds for all $\varphi \in C_0^\infty(U \cap \bar{M}) \cap \mathscr{D}$, then there exists an operator $N\colon L_2^{0,1}(M) \to L_2^{0,1}(M)$ with the following properties, where $L_2^{0,1}(M)$ stands for the forms of type* (0, 1) *with square integrable components.*

(*A*) *N is bonnded, self-adjoint and completely continuous.*
(*B*) *Let $\mathscr{H}^{0,1}$ denote the null space of N. Then*

$$\mathscr{H}^{0,1} = \{\varphi \in \text{Dom}(\text{clos}\,\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \mid \text{clos}\,\bar{\partial}\varphi = 0 \text{ and } \bar{\partial}^*\varphi = 0\},$$

where clos $\bar{\partial}$ *denotes the L_2-closure of $\bar{\partial}$, and $\bar{\partial}^*$ its L_2-adjoint. Frurther $\mathscr{H}^{0,1}$ is finite dimensional.*

(*C*) *Let* $\square = (\text{clos}\,\bar{\partial})\bar{\partial}^* + \bar{\partial}^*(\text{clos}\,\bar{\partial})$, *where the $\bar{\partial}$ in the second term denotes the map from functions into* $(0, 1)$*-forms. Then the range of N is contained in* Dom $(\square)$, *and we have for every $\varphi \in L_2^{0,1}(M)$ the orthogonal decomposition*:

$$(6.4) \qquad \varphi = \square N\varphi + H^{0,1}\varphi,$$

where $H^{0,1}\colon L_2^{0,1}(\bar{M}) \to \mathscr{H}^{0,1}$ is the orthogonal projection onto $\mathscr{H}^{0,1}$. Furthermore $\bar{\partial}\bar{\partial}^*N\varphi$ is orthogonal to $\bar{\partial}^*\bar{\partial}N\varphi$.

(*D*) *N in pseudo-local in the sense that if U is a neighborhood in $\bar{M}$, and $\alpha \in L_2^{0,1}(M)$ such that $\alpha|_U \in C^\infty(U)$, then $N\alpha|_U \in C^\infty(U)$. Furthermore if $\alpha|_U \in H_t(U)$, for $t \geq 0$, then $N\alpha|_U \in H_{t+2s}(U)$.*

(*E*) *Let $\mathscr{H}$ denote the holomorphic functions in $L_2(M)$.* If $\alpha \in$ Dom (clos $\bar{\partial}$), $\bar{\partial}\alpha = 0$ *and* $\alpha \perp \mathscr{H}^{0,1}$, *then there exists a unique* $u \perp \mathscr{H}$ *such that* $\bar{\partial}u = \alpha$. *Thus u may be expressed by*

$$(6.5) \qquad u = \bar{\partial}^*N\alpha .$$

Further, by (D), $\bar{\partial}^*N$ is pseudo-local, and if $\alpha|_U \in H_t(U)$ then $u|_U \in H_{t+s}(U)$.

(*F*) *Let $H\colon L_2(M) \to \mathscr{H}$ be the orthogonal projection onto $\mathscr{H}$. If f is a fuction in* Dom (clos $\bar{\partial}$), *then we have*

$$(6.6) \qquad Hf = f - \bar{\partial}^*N\bar{\partial}f .$$

Again, by (D), H is pseudo-local, and if $f|_U \in H_t(U)$ then $Hf|_U \in H_t(U)$.

In [4] Hörmander proved theorems which do not assume smoothness of bM. He considered a pseudo-convex M in the sense that there exists a real-valued function σ on M such that for each real number c the set $M_c = \{z \in M \mid \sigma(z) \leq c\}$ is compact and such that for $c \geq c_0$ the form $\partial\bar{\partial}\sigma$ restricted to holomorphic vectors tangential to bM_c is positive definite. For such M he showed that if every point has a neighborhood U, α is a $(0, 1)$-form with $\alpha|_U \in L_2^{0,1}(U)$, $\bar{\partial}\alpha = 0$, and there exists $v \in L_2(M_{c_0})$ such that $\bar{\partial}v = \alpha$ in M_{c_0}, then there exists u such that $\bar{\partial}u = \alpha$ in M, and u is locally in L_2. Combining the methods of [11] and [4] and using Theorem 5.14, we obtain the following result.

6.7. Theorem. *If M is pseudo-convex, $P \in bM$ has a neighborhood U such that $U \cap bM$ is smooth and each $Q \in U \cap bM$ is of finite type, α is a* $(0, 1)$*-form, which is locally in L_2 and is C^∞ on $V \cap \bar{M}$ (V open), and $\alpha = \bar{\partial}v$ where v is a function on M which is locally in L_2, then there exists a function u on*

M, locally in L_2, *such that u is* C^∞ $U \cap V \cap \overline{M}$ *and* $\bar\partial u = \alpha$.

As a further consequence of the estimate (6.1) we mention a theorem of Kerzman [7]. Kerzman's proof was intended for the case $s = 1/2$, but his arguments use only the formula (6.6) and the pseudo-locality of N.

6.8. Theorem. *If N is pseudo-local, and H satisfies* (6.6), *then the Bergman kernel function* $K\colon M \times M \to C$ *defined by*

$$Hf(z) = \int_M K(z, \overline{w})\overline{f(w)}dV_w$$

is in $C^\infty(\overline{M} \times M - G)$, *where* $G = \{(z, w) \in (bM) \times (b\overline{M}) \mid z = w\}$.

The following theorem gives a solution to the global Levi problem for two dimensional pseudo-convex manifolds in the same way as is done for strongly pseudo-convex manifolds in [8].

6.9. Theorem. *If M is a pseudo-convex manifold of strict type m, and* $P \in bM$, *then there exists a holomorphic function f on M, which tends to* ∞ *at P but is* C^∞ *on* $\overline{M} - \{P\}$.

Proof. Let h be a holomorphic function in a neighborhood U of P such that (3.2) holds, and let $\zeta \in C_0^\infty$ such that $\zeta = 1$ in a neighborhood V of P. Let α be the (0, 1)-form defined by

$$\alpha = \begin{cases} \bar\partial(\zeta/h) & \text{in } U \cap \overline{M} - \{P\}\,, \\ 0 & \text{in } (\overline{M} - U \cap \overline{M}) \cup \{P\}\,. \end{cases} \tag{6.10}$$

Note that $\alpha \in C^\infty(\overline{M})$ since it vanishes in a neighborhood of P. By choosing ζ with sufficiently small support we conclude, from a result in [4], that there exists $v \in L_2(M)$ such that $\alpha = \bar\partial v$ so that $\alpha \perp \mathscr{H}^{0,1}$; hence by (6.5) we have $\alpha = \bar\partial u$ with $u \in C^\infty(\overline{M})$. Then f defined by

$$f = \begin{cases} u - \zeta/h & \text{in } U \cap \overline{M} - \{P\}\,, \\ u & \text{in } \overline{M} - U \cap \overline{M} \end{cases}$$

has the desired properties.

7. Remarks and open problems

In this section we sketch some extensions of the previous material and formulate some natural questions which arise.

I. *Generalizations to n dimensions*

If M is an n-dimensional complex manifold with smooth boundary bM, and $P \in bM$, then we let $L_1, \cdots, L_n$ be a basis for the holomorphic vector fields in a neighborhood of bM such that $L_n(r) = 1$ and $L_j(r) = 0$ for $1 \leq j \leq n - 1$. The Levi form is then the hermitian $(n - 1) \times (n - 1)$ matrix c_{ij} given by

$$c_{ij} = \langle \partial\bar{\partial}r, L_i \wedge \bar{L}_j \rangle\,, \qquad i, j \leq n-1\,. \tag{7.1}$$

This can also be expressed by

$$[L_i, L_j] = c_{ij}(L_n - \bar{L}_n) + \sum_{k=1}^{n-1} a_{ij}^k L_k + \sum_{k=1}^{n-1} b_{ij}^k \bar{L}_k\,. \tag{7.2}$$

The proof of the following theorem is a straightforward generalization of the proof of Theorem 5.14.

7.3. Theorem. *If M is a pseudo-convex n-dimensional complex manifold with smooth boundary, P has a neighborhood U such that on U there is a basis $L_1, \cdots, L_{n-1}$ of the holomorphic vector fields tangent to bM such that the matrix c_{ij} is diagonal, and there exist integers m_k, $k = 1, n-1$, such that*

$$\langle \partial r, L_k^{(i_0, \cdots, m_k)} \rangle \neq 0 \qquad \text{in } U\,, \tag{7.4}$$

then there exists $C > 0$ such that

$$\sum_{k=1}^{n-1} |||D\varphi_k|||_{2-m_k-1}^2 + \|\varphi_n\|_1^2 \leq C(\|\bar{\partial}\varphi\|^2 + \|\vartheta\varphi\|^2 + \|\varphi\|^2) \tag{7.5}$$

for all $\varphi = \sum_{k=1}^{1} \varphi_k d\bar{z}^k$ with $\varphi_k \in C_0^\infty(\bar{M} \cap U)$ and $\varphi_n|_{U \cap bM} = 0$.

The assumption of diagonalizability of the Levi form is very unnatural and restrictive. The problem of what are the right conditions for sub-elliptic estimates is open. The condition, being of type m, is very analogous to the conditions of Nirenberg and Treves [13] and of Treves [15]. In these papers Nirenberg and Treves deal with the case when all the commutators vanish, and a very important question is how to generalize our results to that case.

II. *Forms of type (p, q)*

The generalization of our results in § 4 to forms of type (p, q) when $n = 2$ and $q > 0$ is immediate.

If θ is a form of type $(1, 1)$:

$$\theta = \sum \theta_{ij} \omega^i \wedge \bar{\omega}^j\,,$$

then the boundary conditions is $\theta_{in} = 0$ on bM, and for pseudo-convex M the estimate

$$\begin{aligned} \sum_i \int_{bM} \lambda^{10} |\theta_{i1}|^2 ds + \sum_i \|L_1\theta_{i1}\|^2 + \sum_{i,j,k} \|\bar{L}_k \theta_{ij}\|^2 \\ \leq C(\|\bar{\partial}\theta\|^2 + \|\vartheta\theta\|^2 + \|\theta\|^2) \end{aligned} \tag{7.6}$$

is obtained in the same way as (4.5) and the rest of the proof follows exactly as in § 4. For forms of type (p, n) the boundary condition is vanishing on the boundary, and hence in that case the $\bar{\partial}$-Neumann problem is coercive and all the first derivatives can then be estimated without any assumptions of the pseudo-convexity. In the case of forms of type (p, q) with $n > 2$, and $0 < q < n$, Theorem 7.3 has an obvious generalization.

III. *Peak points*

A point $P \in bM$ is called a *local peak point* of M if there exist a neighborhood U of P and a holomorphic function f defined on U such that $|f(P)| = 1$ and $|f(Q)| < 1$ for $Q \in U \cap \bar{M}$ and $Q \neq P$. It is clear that P is a peak point if there exists a holomorphic function h in a neighborhood U of P such that

$$\{Q \in U \cap \bar{M} \mid \operatorname{Re} h(Q) = 0\} = \{P\} . \tag{7.7}$$

We can take $f = e^h$. Two questions arise. First, what are the conditions for the existence of a function satisfying (7.7)? Second, what are the conditions for P to be a peak point?

IV. *Precise sub-elliptic estimates*

When $m > 1$ the estimate (5.15) can be improved as follows:

$$|||D\varphi|||_{s-1}^2 \leq C_s(\|\bar{\partial}\varphi\|^2 + \|\vartheta\varphi\|^2 + \|\varphi\|^2) , \tag{7.8}$$

where $s < 1/(m+1)$. The proof of this uses a result of Hörmander [5, Theorem 4.3]. It suffices to show that in the proof of Lemma 5.9 the ε in (5.10) can be taken to be less than $1/(m+1)$.

7.9. Theorem (*Hörmander*). *If $X_1, \cdots, X_k$ are real vector C^∞-fields in a neighborhood of $0 \in \mathbf{R}^n$, $\mathscr{L}_0(X_1, \cdots, X_k)$ is the module over C^∞ complex-valued function spanned by $X_1, \cdots, X_k$, $\mathscr{L}_s(X_1, \cdots, X_k)$ is the module spanned by $\mathscr{L}_{s-1}(X_1, \cdots, X_k)$ and $[\mathscr{L}_{s-1}(X_1, \cdots, X_k), \mathscr{L}_0(X_1, \cdots, X_k)]$, and every C^∞ vector field belongs to $\mathscr{L}_m(X_1, \cdots, X_k)$, then for each $s < 1/(m+1)$ there exist a neighborhood U of 0 and a constant $C_s > 0$ such that*

$$\|u\|_s^2 \leq C_s\left(\sum_{j=1}^k \|X_j u\|^2 + \|u\|^2\right) , \tag{7.10}$$

for all $u \in C_0^\infty(U)$.

To prove (5.10) with $\varepsilon < 1/(m+1)$ it suffices to show that there exist constants δ and C_ε such that

$$\|u(\cdot, r)\|_\varepsilon^2 \leq C_\varepsilon(\|L_1 u(\cdot, r)\|^2 + \|\bar{L}_1 u(\cdot, r)\|^2 + \|u(\cdot, r)\|^2 \tag{7.11}$$

for $-\delta \leq r \leq 0, u \in C_0^\infty(U \cap \bar{M})$, where C_ε is independent of r. Here the norms are taken over the 3-dimensional "slices" $(\cdot, r) = (x_1, x_2, x_3, r)$ with r fixed. First observe that (7.11) implies (5.10) since by integrating with regard to r we obtain

$$|||u|||_\varepsilon^2 \le C_\varepsilon(\|L_1u\|^2 + \|\bar{L}_1u\|^2 + \|u\|^2) , \tag{7.12}$$

if we assume that U is sufficiently small so that it is contained in the strip $-\delta \le r \le 0$. Then (5.10) follows since (as proven in Lemma 5.9)

$$|||Du|||_{\varepsilon-1}^2 \le \text{const. } (|||u|||_\varepsilon^2 + \|L_1u\|^2 + \|\bar{L}_1u\|^2 + \|\bar{L}_2u\|^2 + \|u\|^2) . \tag{7.13}$$

To conclude we show that (7.11) follows from (7.10) with $\varepsilon < 1/(m+1)$. Set $L_1 = X_1 + iX_2$, where X_1 and X_2 are real vector fields. Since 0 is a point of type m, $\mathscr{L}_m(X_1, X_2)$ contains all the vector fields. Further we have

$$\sum_{j=1}^{2} \|X_j u(\cdot, r)\|^2 \le \text{const. } (\|L_1u(\cdot, r)\|^2 + \|\bar{L}_1u(\cdot, r)\|^2 + \|u(\cdot, r)\|^2) . \tag{7.14}$$

So (7.11) follows, and the fact that C_ε may be chosen independent of r follows from the proof of Theorem 7.9.

The above estimate makes plausible the conjecture that we can actually take $\varepsilon = 1/(m+1)$ as is the case for $m = 1$. Further it seems plausible that for a point of type m the estimate does not hold for $\varepsilon > 1/(m+1)$.

V. *Hölder estimates*

Recently there has been a great deal of interest in solving the equation $\bar\partial u = \alpha$ for α bounded with u bounded and satisfying Hölder estimates [2], [3], [6]. In [4] Kerzman showed that on strongly pseudo-convex manifolds a solution u exists satisfying Hölder estimates of order $s < 1/2$. In a letter Henkin and Romanoff announced the improvement of this result to $s = 1/2$. Thus two natural questions arise. First, does the unique solution, orthogonal to holomorphic functions satisfy the Hölder estimates? Second, in case M is pseudo-convex and of type m, can one find a solution satisfying Hölder estimates, in particular, estimates of order $1/(m+1)$?

The second question seems to be intimately related with the existence of peak functions and on precise information about these functions.

VI. *Regularity of the Bergman kernel function*

Recalling Theorem 6.8, we know that for a pseudo-convex M of type m the kernel function K is in $C^\infty(\bar{M} \times \bar{M} - G)$ where G is the diagonal of $(bM) \times (bM)$. The question is what happens to K near G. In [4] Hörmander showed that $K(z, \bar{z})$ behaves like $|z - P|^{-n-1}$ if $P \in bM$ is a strongly pseudo-convex point (i.e., of type 1). If P is of type m, it seems conceivable that near P, $K(z, \bar{z})$ behaves like $|z - P|^{-n-1/m}$. Again the answer to this question is closely related to the existence and properties of a function satisfying (7.7).

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